

Diversity-Multiplexing Tradeoff of Cooperative Communication with Linear Network Coded Relays

Hakan Topakkaya and Zhengdao Wang

Dept. of Elec. and Comp. Eng., Iowa State University, Ames, IA 50011-3060

Emails: {hakan, zhengdao}@iastate.edu

Abstract—Network coding and cooperative communication have received considerable attention from the research community recently in order to mitigate the adverse effects of fading in wireless transmissions and at the same time to achieve high throughput and better spectral efficiency. In this work, we analyze a network coding scheme for a cooperative communication setup with multiple sources and destinations. The proposed protocol achieves the full diversity order at the expense of a slightly reduced multiplexing rate compared to existing schemes in the literature. We show that our scheme outperforms conventional cooperation in terms of the diversity-multiplexing tradeoff.

Index Terms—Cooperative communication, network coding, random network coding, outage probability, diversity-multiplexing tradeoff.

I. INTRODUCTION

Channel fading is one significant cause of performance degradation in wireless networks. In order to combat fading, diversity techniques that operate in time, frequency or space are commonly employed. The basic idea is to send the signals that carry same information through different paths, allowing the receiver to obtain multiple independently faded replicas of the data symbols. Cooperative diversity tries to exploit spatial diversity using a collection of distributed antennas belonging to different terminals, hence creating a virtual array rather than using physical arrays.

In [1] Ahlswede et al. introduced *network coding* to achieve the max-flow value for single-source multicast which could be impossible by simply routing the data. Since then, network coding has been recognized as a useful technique in increasing the throughput of a wired/wireless network. The basic idea of network coding is that an intermediate node does not simply route the information but instead combines several input packets from its neighbors with its own packets and then forwards it to the next hop. However, since network coding is devised at the network layer, error-free communication from the physical and medium-access layer is usually assumed, which is a simplifying assumption for wireless communications.

Efforts have also been made to apply network coding to the physical layer, e.g. in [9]. Towards that goal, cooperative schemes have been proposed that make use of network coding in a cooperative communication setup, and studies have been conducted to determine whether network coding provides any advantages over existing cooperative communication techniques [6], [8].

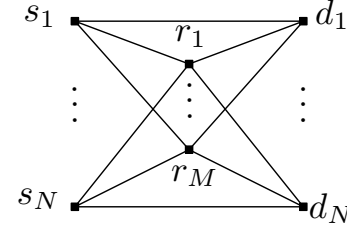


Fig. 1. System model: N source-destination pairs and M relays

In [6], a network-coded cooperation (NCC) was proposed and its performance was quantified using the diversity-multiplexing tradeoff analysis which was originally proposed for multiple antenna systems in [10]. NCC was shown to outperform conventional cooperation (CC) schemes which includes space-time coded protocols [4] and selection relaying [2]. NCC requires less bandwidth, and yields similar or reduced system outage probability while achieving the same diversity order. However, these results are based on an optimistic assumption that any destination node should receive the packets that are not intended for it without any error so that the intended packet can be recovered from the xor'ed packet sent by the relay. When this assumption is removed the scheme can no longer achieve the full diversity order of $M + 1$, where M is the number of cooperating relays, but only a reduced diversity order of 2.

In this paper, we propose a network coded cooperation schemes for N source-destination pairs assisted with M relays. The proposed scheme allows the relays to apply network coding on the data it has received from its neighbors using the coefficients from the parity-check matrix of a $(N + M, M, N + 1)$ MDS code. A closed form expression for the outage probability is derived. We also obtain the diversity-multiplexing tradeoff performance of the proposed scheme. Specifically, it achieves the full-diversity order $M + 1$ at the expense of a slightly reduced multiplexing rate. We show that our scheme outperforms NCC and CC in terms of the probability of outage.

The rest of the paper is organized as follows. Section II discusses the system model, description of the proposed scheme and the design of the network code. In Section III, we present the main result and the proof. In Section IV the performance of the proposed scheme is compared in terms of DMT and average outage probability. Section V contains the conclusions.

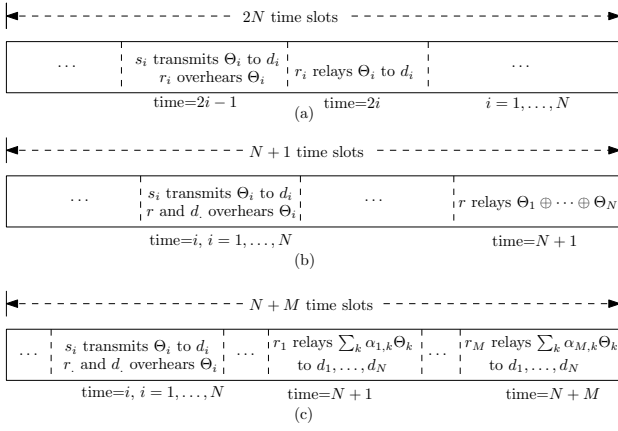


Fig. 2. time-division allocation for the different schemes compared: (a) CC (b) NCC (c) DNCC

II. SYSTEM MODEL

A. General System Description

The network studied in the paper is composed of N source-destination pairs denoted as $(s_1, d_1), \dots, (s_N, d_N)$, and M relays denoted as r_1, \dots, r_M in a single-cell where all the nodes can hear the transmissions of each other as shown in Fig. 1. We consider two different transmission scenarios. In the first scenario, each source node s_i is trying to transmit the data packet θ_i to all the destinations $d_i, i = 1, \dots, N$ which is known as the *multicast* scenario. In the second scenario, each source node s_i is trying to transmit the data packet θ_i to only destination d_i and we will refer to this scenario as the *unicast* scenario. We assume that each packet is composed of L bits: $\theta_i = [\theta_{i,1}, \theta_{i,2}, \dots, \theta_{i,L}]$. As in [3], these length- L blocks of bits that are transmitted on each link will be treated as elements of a finite field $\mathbb{F}_q, q = 2^L$. All the nodes are assumed to be equipped with half-duplex (i.e. cannot transmit and receive at the same frequency) single-antennas. Each data packet Θ_i is error control coded and modulated, and transmitted in T time slots.

We assume the users transmit in equal time slots with transmit power \bar{P} . The channel between any pair of nodes is assumed to be frequency flat fading with additive white Gaussian noise (AWGN). Let $\mathbf{x}_j \in \mathbb{C}^{1 \times T}$ denote the transmitted symbols from node j and $\mathbf{y}_i \in \mathbb{C}^{1 \times T}$ the received symbols at node i , and the additive noise has independent and identically distributed (i.i.d.) entries $\mathbf{z}_{i,t} \sim \mathcal{CN}(0, 1)$, and $\mathbf{h}_{i,j} \in \mathbb{C}$ the instantaneous channel realization. Then, the channel within one block can be written as

$$\mathbf{y}_i = \sqrt{\rho} \mathbf{h}_{i,j} \mathbf{x}_j + \mathbf{z}_i \quad (1)$$

where ρ is the average received SNR at the destination. In the above equation, the transmitter could be any of the sources or relays, the receiver could be any of the relays or destinations, as long as the transmitter and receiver are different (i.e., not the same relay). We assume that the channel coefficient $\mathbf{h}_{i,j}$ remains constant during the transmission time of a packet.

The channel coefficient $\mathbf{h}_{i,j}$ between any two nodes is modeled as i.i.d. with zero-mean, circularly symmetric complex Gaussian random variables with common variance $1/\beta$.

Therefore, $|\mathbf{h}_{i,j}|^2$ are exponentially distributed with parameter β . A total of NL bits are transmitted in $(N+M)T$ channel uses, therefore the system rate is $R = \frac{NL}{(N+M)T}$ bits per channel use (BPCU). The transmission rate R_i for one source or one relay per one packet is fixed, identical, and equal to $R_i = \frac{L}{T} = \frac{N+M}{N} R$ BPCU.

B. Deterministic Network Coded Cooperation (DNCC)

Our transmission scheme consists of two stages; see Fig. 2(c). In the first stage, direct transmissions from the sources to the destinations take place in N orthogonal time slots. Thanks to the broadcast nature of the wireless medium, all the destinations and the relays overhear the transmissions. At the end of the first stage, each relay tries to decode all N packets. If a relay can successfully decode all the packets (we denote this assumption by \mathcal{A}), then it participates in the second stage. Otherwise, it remains silent.

In the second stage, the participating relays perform network coding. Specifically, relay i will transmit the linear combination

$$\sum_{k=1}^N \alpha_{ik} \Theta_k \quad (2)$$

where Θ_k is the corresponding finite field element in \mathbb{F}_q for θ_k .

The coefficients α_{ij} 's are predetermined and they are designed in a way to maximize the chance that the received linear combinations are actually decodable at the destination. We discuss the problem of how to choose these predetermined coefficients in detail in Sec. II-C.

In order to express the overall transmitted data packets, we define the following matrix:

$$A := \begin{bmatrix} 1 & 0 & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \ddots & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 1 \\ \alpha_{1,1} & \alpha_{1,2} & \cdot & \cdot & \alpha_{1,N} \\ \alpha_{2,1} & \cdot & \cdot & \cdot & \alpha_{2,N} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_{M,1} & \alpha_{M,2} & \cdot & \cdot & \alpha_{M,N} \end{bmatrix} \quad (3)$$

where the top part is an $N \times N$ identity matrix. We also define the $N \times 1$ finite field vector corresponding to the original source packets as

$$\Theta = [\Theta_1, \Theta_2, \dots, \Theta_N]^T \quad (4)$$

where $(\cdot)^T$ denotes transpose. Using matrices A and Θ , the potential transmitted data packets can be expressed as $X = A\Theta$.

C. Design of the Linear Network Coding Matrix

Each destination d_i will attempt to decode all the packets that have been transmitted at the two stages. It is possible that some of the packets cannot be decoded due to severe channel fading. However, if enough number of packets can be decoded, and the linear network coding coefficients $\alpha_{m,n}$ are

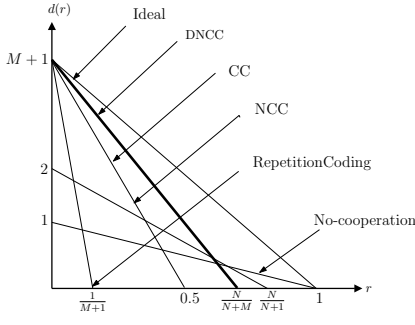


Fig. 3. DMT comparison of various schemes

judiciously designed, then the destination will be able to obtain the transmitted packets by solving a linear system of equations. In this subsection, we discuss the issue of the design of the coding matrix A (or the bottom M rows of A to be exact).

The row Kruskal-rank [7] of A , denoted as $\kappa(A)$, is the number r such that every set of r rows of A is linearly independent, but there exist one set of $r + 1$ rows that are linearly dependent. The column Kruskal-rank can be defined similarly.

The following result relates the column Kruskal-rank of the parity-check matrix of a linear block code to its minimum distance d_{min} : The column Kruskal-rank $\kappa(H)$ of the parity check matrix H of a linear block error-control code and the minimum Hamming distance d_{min} of the code are related by $d_{min} = \kappa(H) + 1$. Please refer to [5] for a proof.

We would like to have the matrix A to have a large row Kruskal rank so that a minimum number of equations is needed by any destination to solve for the source packets. The row Kruskal rank is of course less than or equal to the rank, which is N . To maximize the Kruskal rank, we should design A such that it has Kruskal rank equal to N . Or stated differently, any N rows of the matrix A should be linearly independent. Depending on the sizes N and M , this may or may not be possible in a given finite field \mathbb{F}_q . Note that whether this is possible is closely related whether maximum distance separable (MDS) codes exist in a certain field for certain code dimensions [5].

The transpose H^T of the parity check matrix of a systematic $(N + M, M, N + 1)$ MDS code can be used as an encoding matrix A for our DNCC scheme to minimize the total number of packets necessary at the destinations for decoding the source packets. If such an A is used, then each destination needs and only needs N packets (from the sources and relays) for correct decoding.

III. DIVERSITY-MULTIPLEXING TRADEOFF

As mentioned in the introduction, we will investigate the performance of the proposed scheme via diversity-multiplexing tradeoff (DMT). DMT is widely accepted as a useful performance analysis tool in cooperative systems [2], [6]. For completeness, we give the formal definitions as in [10]: A scheme $C(\rho)$ is said to achieve spatial multiplexing gain r and diversity gain d if the data rate is

$$\lim_{\rho \rightarrow \infty} \frac{R(\rho)}{\log(\rho)} = r, \quad (5)$$

and the average error probability is

$$\lim_{\rho \rightarrow \infty} \frac{\log(P_e(\rho))}{\log(\rho)} = -d. \quad (6)$$

A. Outage Probability

In [10], for long enough block length T it was shown that the probability of error is dominated by the outage probability. Therefore, we will only consider outage probabilities in our analysis. The instantaneous mutual information of the channel model in (1) is given by:

$$I(\mathbf{X}_i; \mathbf{Y}_j) = \log(1 + |\mathbf{h}_{i,j}|^2 \rho). \quad (7)$$

where \mathbf{X}_i and \mathbf{Y}_j denote the transmitted symbol by node i and received symbol by node j . We write $a(\tau) \cong b(\tau)$ if $\lim_{\tau \rightarrow 0} [a(\tau)/b(\tau)] = 1$. An outage event between a transmitter i and receiver j occurs when the instantaneous mutual information for the channel $h_{i,j}$ is less than the transmission rate R_i BPCU. Since $|\mathbf{h}_{i,j}|^2$ is exponentially distributed, the outage probability for the channel in (1) is given by:

$$\begin{aligned} P_0 &= \Pr(I(\mathbf{X}_i; \mathbf{Y}_j) < R_i) \\ &= \Pr(|\mathbf{h}_{i,j}|^2 < \tau) = 1 - \exp(-\beta\tau) \cong \beta\tau, \end{aligned} \quad (8)$$

where $\tau = \frac{2^{\frac{N+M}{N}R_i} - 1}{\rho}$.

B. Further Improvements

1) *Decoding at the relays:* Decode-and-forward schemes suffer from performance loss when the source-relay channel is in outage. And if a multi-source scenario is considered the performance loss becomes even more severe. Therefore, the assumption that the relay has to decode all the packets in order to be able to cooperate becomes a bottleneck for such schemes. We could relax the assumption \mathcal{A} and assume that the relays will participate cooperation even though they have not been able to decode all the packets. Specifically, in the second stage, the participating relays perform network coding on the packets that they have received correctly. If relay i was able to decode the packets correctly from the sources in the set S_i where $S_i \subseteq \{1, \dots, N\}$, then it will transmit the linear combination

$$\sum_{k \in S_i} \alpha_{i,k} \Theta_k \quad (9)$$

where Θ_k is the corresponding finite field element in \mathbb{F}_q for θ_k .

Theorem 1. *The diversity-multiplexing tradeoff of the linear network coded cooperation with M intermediate relay nodes for both multicast and unicast and both either with or without the assumption \mathcal{A} is:*

$$d(r) = (M + 1) \left[1 - \frac{(N + M)}{N} r \right], r \in \left(0, \frac{N}{N + M} \right) \quad (10)$$

Proof: Due to severe fading some of the received packets may not be successfully decoded by the destination. This can be viewed as some of the rows of A being erased. We denote the resulting submatrix by A_i for destination d_i . The

successfully received packets at destination d_i after decoding can be expressed as $Y_i = A_i\Theta$. In the multicast problem, destination d_i cannot recover Θ_i when the submatrix A_i is rank deficient, i.e. when $\text{rank}(A_i) < N$. This happens when less than N channels are not in outage which results in an A_i matrix that has at most $N - 1$ rows. We define this event as $E_{1i} = \{A_i \text{ has at most } N - 1 \text{ rows}\}$.

Next, we define several other events which will simplify the analysis. Let E_m denote the event that m relays fail to receive all the Θ_i 's correctly. Then we have:

$$P(E_m) = \binom{M}{m} P(\varepsilon)^{M-m} (1 - P(\varepsilon))^m$$

$$P(\varepsilon) = \prod_{i=1}^N \Pr(I_{s_i r}(X; Y) > R_i) \quad (11)$$

$$= \prod_{i=1}^N \Pr(|h_{s_i r}|^2 > \tau) = \exp(-N\beta\tau) \quad (12)$$

Similarly, define $E(m, n)$ to be the event that n channels out of m were in outage:

$$P(E(m, n)) = \binom{m}{n} P_0^n (1 - P_0)^{m-n} \quad (13)$$

where P_0 is given by (8). Now, we can express $P(E_{1i})$ as:

$$P(E_{1i}) = \sum_{m=0}^M P(E_m) \cdot \sum_{n=M-m+1}^{N+M-m} P(E(N+M-m, n)) \quad (14)$$

The first summation stands for the probability of the event that m of the relays were in outage, leaving us with only $N+M-m$ nodes which were not in outage for that particular destination. Notice that, as $\rho \rightarrow \infty$, $\tau \rightarrow 0$. We would like to find the following limit:

$$\lim_{\tau \rightarrow 0} \frac{P(E_{1i})}{\tau^{M+1}} \quad (15)$$

We consider the individual terms in the summations one-by-one and find the term with the smallest order of τ . Observe that $P(E_m) \cong K_m \tau^m$ where K_m is a constant:

$$\lim_{\tau \rightarrow 0} \frac{P(E_m)}{\tau^m} = \binom{M}{m} \exp(-N(M-m)\beta\tau)$$

$$\cdot \lim_{\tau \rightarrow 0} \left\{ \frac{(1 - \exp(-N\beta\tau))^m}{\tau^m} \right\} = \binom{M}{m} (N\beta)^m \quad (16)$$

Next, consider the term $P(E(m, n))$,

$$\lim_{\tau \rightarrow 0} \frac{P(E(m, n))}{\tau^n} = \lim_{\tau \rightarrow 0} \frac{\binom{m}{n} P_0^n (1 - P_0)^{m-n}}{\tau^n}$$

$$= \binom{m}{n} \beta^n \quad (17)$$

When n is equal to $M - m + 1$, we have the smallest order τ term. Substituting $n = M - m + 1$ in $P(E(N+M-m, n))$,

we have:

$$\lim_{\tau \rightarrow 0} \left\{ \binom{N+M-m}{M-m+1} (1 - P_0)^{N+M-m-(M-m+1)} \cdot \frac{P_0^{M-m+1}}{\tau^{M-m+1}} \right\} = \binom{N+M-m}{M-m+1} \beta^{M-m+1}$$

Therefore, we have:

$$\lim_{\tau \rightarrow 0} \frac{P(E_m)}{\tau^m} \cdot \frac{P(E(N+M-m, n))}{\tau^{M-m+1}}$$

$$= \binom{M}{m} (N\beta)^m \binom{N+M-m}{M-m+1} \beta^{M-m+1} \quad (18)$$

Finally, using (18), we have:

$$P(E_{1i}) \cong \tau^{M+1} \sum_{m=0}^M \binom{M}{m} N^m \binom{N+M-m}{M-m+1} \beta^{M+1}$$

$$= \left(\frac{2^{\frac{N+M}{N}R-1}}{\rho} \right)^{M+1} \sum_{m=0}^M \binom{M}{m} \binom{N+M-m}{M-m+1} N^m \beta^{M+1} \quad (19)$$

Now, suppose the fixed rate has chosen to be:

$$R = r \log \rho \quad (20)$$

Substituting (20) into (19), we arrive at:

$$P(E_{1i}) \cong \rho^{(\frac{N+M}{N}r-1)(M+1)}$$

$$\cdot \sum_{m=0}^M \binom{M}{m} \binom{N+M-m}{M-m+1} N^m \beta^{M+1} \quad (21)$$

Since the summation term is just a constant in terms of ρ , we have the desired result. We skip the proof for the unicast and without assumption \mathcal{A} cases due to space limitations. ■

IV. COMPARISON WITH OTHER SCHEMES

A. DMT Comparison

In this section, we would like to compare the proposed scheme with the existing schemes in the literature in terms of diversity-multiplexing tradeoff. The closest scheme in the literature is the Network-coded cooperation (NCC) considered in [6]. There are N s-d pairs and M relays in a single cell. However, instead of all the relays, only one relay transmits following the direct transmissions from the source nodes, which results in total of $N+1$ time slots (Fig. 2 (b)). Using fewer time-slots NCC achieves a better spectral efficiency than DNCC. However, NCC can only provide a fixed diversity order of 2, while the proposed scheme achieves the full-diversity order of $M+1$.

In the following, for comparison we include the DMT performance of NCC and that of conventional cooperation (CC).

The diversity-multiplexing tradeoff of NCC is given by [6]:

$$d(r) = 2 \left(1 - \frac{N+1}{N} r \right), \quad r \in \left(0, \frac{N}{N+1} \right) \quad (22)$$

The DMT of the CC schemes with M intermediate relay nodes is given by [2], [4]:

$$d(r) = (M+1)(1-2r), \quad r \in (0, 0.5). \quad (23)$$

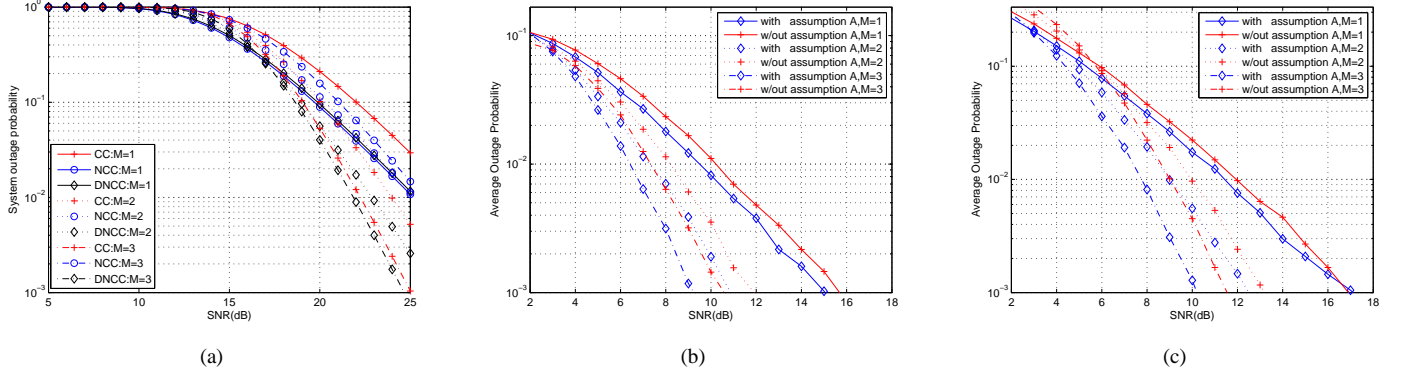


Fig. 4. System outage probability ($N=2, M=1, \dots, 3$), Average outage probability per destination, ($N=2, M=1, \dots, 3$ and $N=3, M=1, \dots, 3$)

We present diversity-multiplexing tradeoff of the existing schemes and DNCC in Fig. 3. As can be seen from the figure, although DNCC can achieve the full diversity order, NCC fails to do so. We also see that DNCC achieves a better DMT than CC when $N > M$.

B. System Outage Probability

Here, we compare the system outage probability of DNCC with the other schemes. The system outage occurs when *any* d_i is unable to decode Θ_i reliably:

$$P_s = 1 - \prod_{i=1}^N (1 - P(E_{1i})) \quad (24)$$

We compare (24) with the system outage probabilities (30), (43) derived in [6]. In Fig. 4(a) the proposed method clearly outperforms NCC by achieving the full diversity of $M+1$ as compared to NCC's fixed diversity order of 2.

C. Monte-Carlo Simulation

We also compare the proposed scheme with the existing schemes via Monte-Carlo simulations. In the simulations, only channel conditions are considered to isolate the diversity benefits of the scheme. We generate an $(N+M) \times N$ and an $N \times N$ matrix that contains the channel coefficients for each destination and each relay, respectively. Then, we decide that the transmission is successful for any link if the instantaneous channel condition is large enough to be able to support the given data rate and we update the same size linear coefficient matrices accordingly. After all the transmissions take place, we perform Gaussian elimination on the updated linear coefficient matrices to conclude whether each destination d_i was able to recover the source packet Θ_i or not. The channel coefficient variances are chosen to be equal to one and $R_i = 1$ BPCU. Please note that we considered the average outage error probability which is found by dividing the total number of errors occurred by the number of source nodes instead of the system error probability. In all the figures only the unicast scenario is adapted since CC cannot be implemented in a multicast scenario. As it is proven by the Theorem 1, the performance loss incurred due to the assumption \mathcal{A} is not in terms of diversity gain but it is in terms of coding gain. This

is validated through simulations as shown in the Fig. 4(b) and Fig. 4(c).

V. CONCLUSIONS

In this paper, we have proposed a network coded cooperation scheme for N source-destination pairs assisted with M relays. We studied two different transmission scenarios, unicast and multicast. The proposed scheme allows the relays to apply network coding on the data it has received from its neighbors. We establish the link between the parity-check matrix for a $(N+M, M, N+1)$ MDS code and the coefficients to perform network coding in a cooperative communication scenario consisting of N source-destination pairs and M relays. We also obtained the diversity-multiplexing tradeoff performance of the proposed scheme, and showed its advantage over the existing schemes. Specifically, it achieves the full-diversity order $M+1$ at the expense of a slightly reduced multiplexing rate.

REFERENCES

- [1] R. Ahlswede, N. Cai, S.-Y. R. Li, and R. W. Yeung, "Network information flow," *IEEE Trans. Inform. Theory*, vol. 46, no. 4, pp. 1204–1216, Apr. 2000.
- [2] A. Bletsas, A. Khisti, D. P. Reed, and A. Lippman, "A simple cooperative diversity method based on network path selection," *IEEE Journal on Selected Areas in Communications*, vol. 24, no. 3, pp. 659–672, Mar. 2006.
- [3] R. Koetter and M. Medard, "An algebraic approach to network coding," *IEEE/ACM Transactions on Networking*, vol. 11, no. 5, pp. 782–795, Oct. 2003.
- [4] J. N. Laneman and G. W. Wornell, "Distributed space-time-coded protocols for exploiting cooperative diversity in wireless networks," *IEEE Trans. Inform. Theory*, vol. 49, no. 10, pp. 2415–2425, Oct. 2003.
- [5] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error-Correcting Codes*, Amsterdam: North-Holland, 1977.
- [6] C. Peng, Q. Zhang, M. Zhao, Y. Yao, and W. Jia, "On the performance analysis of network-coded cooperation in wireless networks," *IEEE Transactions on Wireless Communications*, vol. 7, no. 8, pp. 3090–3097, August 2008.
- [7] Z. Wang and G. B. Giannakis, "Complex-field coding for OFDM over fading wireless channels," *IEEE Trans. Inform. Theory*, vol. 49, no. 3, pp. 707–720, Mar. 2003.
- [8] M. Yu, J. Li, and R. S. Blum, "User cooperation through network coding," *Proc. of IEEE Intl. Conf. on Com.*, pp. 4064–4069, June 2007.
- [9] S. Zhang, S. Liew, and P. Lam, "Physical layer network coding," *Annual International Conference on Mobile Computing and Networking (ACM Mobicom)*, Sept. 2006.
- [10] L. Zheng and D. N. C. Tse, "Diversity and multiplexing: a fundamental tradeoff in multiple-antenna channels," *IEEE Trans. Inform. Theory*, vol. 49, no. 5, pp. 1073–1096, May 2003.